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Bäcklund and Darboux Transformations. The Geometry of Solitons

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This book is devoted to a classical topic that has undergone rapid and fruitful development over the past 25 years, namely Bäcklund and Darboux transformations and their applications in the theory of integrable systems, also known as soliton theory.

The book consists of two parts. The first is a series of introductory pedagogical lectures presented by leading experts in the field. They are devoted respectively to Bäcklund transformations of Painlevé equations, to the dressing method and Bäcklund and Darboux transformations, and to the classical geometry of Bäcklund transformations and their applications to soliton theory. The second part contains original contributions that represent new developments in the theory and applications of these transformations.

Both the introductory lectures and the original talks were presented at an International Workshop that took place in Halifax, Nova Scotia (Canada). This volume covers virtually all recent developments in the theory and applications of Bäcklund and Darboux transformations.

Titles in this series are co-published with the Centre de Recherches Mathématiques.

✧ Readership

Graduate students and research mathematicians interested in dynamical systems, ergodic theory, and partial differential equations.

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Multi-Component Matrix KP Hierarchies as Symmetry-Enhanced Scalar KP Hierarchies and Their Darboux-Bäcklund Solutions

H. Aratyn¹, E. Nissimov^{2,3} and S. Pacheva^{2,3}

Abstract

We show that any multi-component matrix KP hierarchy is equivalent to the standard one-component (scalar) KP hierarchy endowed with a special infinite set of abelian additional symmetries, generated by squared eigenfunction potentials. This allows to employ a special version of the familiar Darboux-Bäcklund transformation techniques within the ordinary scalar KP hierarchy in the Sato formulation for a systematic derivation of explicit *multiple*-Wronskian tau-function solutions of all multi-component matrix KP hierarchies.

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1. Introduction. Background on the KP Hierarchy and Ghosts Symmetries.

Multi-component generalizations of Kadomtsev-Petviashvili (KP) hierarchy of integrable nonlinear soliton equations [1] attract a lot of interest both from physical and mathematical point of view. They are known to contain such physically relevant nonlinear integrable systems as Davey-Stewartson, two-dimensional Toda lattice and three-wave resonant interaction ones [2]. On the other hand, multi-component KP hierarchies turn out to be intimately connected to classical geometry of conjugate nets and the classification problem of Hamiltonian systems of hydrodynamical type [3].

There exist several equivalent formulations of multi-component KP hierarchies: matrix pseudo-differential operator (Sato) formulation; tau-function approach via matrix Hirota bilinear identities; multi-component free fermion formulation. Here we will offer yet another approach. Namely, we will show that any multi-component $N \times N$ matrix KP hierarchy [1] is equivalent to the standard one-component (scalar) KP hierarchy endowed with $N - 1$ copies of mutually commuting infinite-dimensional algebras of abelian additional (“ghost”) symmetries, generated by squared eigenfunction potentials of the initial hierarchy (see definition in (7) below). The latter “ghost” symmetry enhanced scalar KP hierarchy will be called “multiple-KP hierarchy”. The principal advantage of multiple-KP hierarchy formulation over the standard Sato matrix pseudo-differential operator formulation lies in the fact that the former allows to use a special version of the well-known Darboux-Bäcklund (DB) transformation techniques within the ordinary scalar KP hierarchy in the Sato formulation for a systematic derivation of soliton-like *multiple*-Wronskian tau-function solutions of the multi-component KP hierarchies (Section 5 below).

The starting point of our presentation is the pseudo-differential Lax operator \mathcal{L} of scalar KP hierarchy obeying KP evolution equations w.r.t. the multi-time $(t) \equiv (t_1 \equiv x, t_2, \dots)$ (for notations and review, see [4]) :

$$\mathcal{L} = D + \sum_{i=1}^{\infty} u_i D^{-i} \quad ; \quad \frac{\partial \mathcal{L}}{\partial t_l} = \left[(\mathcal{L}^l)_+, \mathcal{L} \right] \quad , \quad l = 1, 2, \dots \quad (1)$$

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The symbol D stands for the differential operator $\partial/\partial x$, whereas $\partial \equiv \partial_x$ will denote derivative of a function. The subscripts (\pm) indicate purely differential/pseudo-differential part of the corresponding operator. Equivalently, one can represent Eq.(1) in terms of the dressing operator W whose pseudo-differential series are expressed in terms of the so called tau-function $\tau(t)$:

$$\mathcal{L} = WDW^{-1} \quad , \quad \frac{\partial W}{\partial t_l} = -(\mathcal{L}^l)_- W \quad , \quad W = \sum_{n=0}^{\infty} \frac{p_n(-[\partial])\tau(t)}{\tau(t)} D^{-n} \quad (2)$$

with the notation: $[y] \equiv (y_1, y_2/2, y_3/3, \dots)$ for any multi-variable $(y) \equiv (y_1, y_2, y_3, \dots)$, in particular $(\partial) \equiv (\partial/\partial t_1, \partial/\partial t_2, \dots)$, and with $p_k(y)$ being the Schur polynomials. The tau-function is related to the Lax operator as (below ‘‘Res’’ denotes the coefficient in front of D^{-1}) :

$$\partial_x \frac{\partial}{\partial t_l} \ln \tau(t) = \text{Res} \mathcal{L}^l \quad , \quad \tau(t) \longrightarrow \tau(t) e^{\sum_{l=1}^{\infty} c_l t_l} \quad (3)$$

The second relation (3) tells that $\tau(t)$ is defined up to an exponential linear w.r.t. (t) .

In the present approach a basic notion is that of (adjoint) eigenfunctions (EF’s) $\Phi(t), \Psi(t)$ of the scalar KP hierarchy satisfying :

$$\frac{\partial \Phi}{\partial t_k} = \mathcal{L}_+^k(\Phi) \quad ; \quad \frac{\partial \Psi}{\partial t_k} = -(\mathcal{L}^*)_+^k(\Psi) \quad (4)$$

Throughout this paper we will rely on an important tool provided by the spectral representation of EF’s [5]. The spectral representation is equivalent to the following statement: Φ and Ψ are (adjoint) EF’s if and only if they obey the integral representation:

$$\Phi(t) = \int dz \frac{e^{\xi(t-t',z)} \tau(t - [z^{-1}]) \tau(t' + [z^{-1}])}{z \tau(t) \tau(t')} \Phi(t' + [z^{-1}]) \quad (5)$$

$$\Psi(t) = \int dz \frac{e^{\xi(t'-t,z)} \tau(t + [z^{-1}]) \tau(t' - [z^{-1}])}{z \tau(t) \tau(t')} \Psi(t' - [z^{-1}]) \quad (6)$$

where $\int dz$ denotes normalized contour integral around origin.

Further crucial notion to be employed in the present construction is the so called squared eigenfunction potential (SEP) $S(\Phi, \Psi)$ of arbitrary pair of an EF Φ and an adjoint EF Ψ [9] (cf. also [5]) :

$$S(\Phi, \Psi) = \partial^{-1}(\Phi \Psi) \quad , \quad \frac{\partial}{\partial t_n} S(\Phi(t), \Psi(t)) = \text{Res} \left(D^{-1} \Psi (\mathcal{L}^n)_+ \Phi D^{-1} \right) \quad , \quad n = 1, 2, \dots \quad (7)$$

The flow equations above fix the ambiguity in applying the inverse derivative ∂^{-1} in the SEP definition up to an overall trivial constant. In what follows ∂^{-1} will appear always in the form of SEP’s (first Eq.(7)).

Consider now an infinite system of independent (adjoint) EF’s $\{\Phi_j, \Psi_j\}_{j=1}^{\infty}$ of the standard one-component KP hierarchy Lax operator \mathcal{L} and define the following infinite set of the additional ‘‘ghost’’ symmetry flows [6]:

$$\frac{\partial}{\partial t_s} \mathcal{L} = [\mathcal{M}_s, \mathcal{L}] \quad , \quad \mathcal{M}_s = \sum_{j=1}^s \Phi_{s-j+1} D^{-1} \Psi_j \quad (8)$$

$$\frac{\partial}{\partial t_s} \Phi_k = \sum_{j=1}^s \Phi_{s-j+1} \partial^{-1}(\Psi_j \Phi_k) - \Phi_{k+s} \quad ; \quad \frac{\partial}{\partial t_s} \Psi_k = \sum_{j=1}^s \Psi_j \partial^{-1}(\Phi_{s-j+1} \Psi_k) + \Psi_{k+s} \quad (9)$$

$$\frac{\partial}{\partial \bar{t}_s} F = \sum_{j=1}^s \Phi_{s-j+1} \partial^{-1}(F \Psi_j) \quad ; \quad \frac{\partial}{\partial \bar{t}_s} F^* = \sum_{j=1}^s \Psi_j \partial^{-1}(\Phi_{s-j+1} F^*) \quad (10)$$

where $s, k = 1, 2, \dots$, and F, F^* denote generic (adjoint) EF's which do not belong to the “ghost” symmetry generating set $\{\Phi_j, \Psi_j\}_{j=1}^\infty$.

Transformations in (8)-(10) contain terms in $a\partial^{-1}b$, whose study was initiated in [7] and exploited in [8] in which inhomogeneous terms in symmetry transformations appeared in the particular case of constrained KP hierarchies.

It is easy to show that the “ghost” symmetry flows $\partial/\partial \bar{t}_s$ from Eqs.(8)-(10) commute, namely, that the ∂ -pseudo-differential operators \mathcal{M}_s (8) satisfy the zero-curvature equations: $\partial \mathcal{M}_r / \partial \bar{t}_s - \partial \mathcal{M}_s / \partial \bar{t}_r - [\mathcal{M}_s, \mathcal{M}_r] = 0$.

Let us consider the (adjoint)-EF Eqs.(4) for $k=2$ and $\Phi = \Phi_1, \Psi = \Psi_1$ (the pair generating the lowest $s=1$ “ghost” symmetry flows (8)–(10)) together with their $s=2$ “ghost” symmetry flow Eqs.(9) (i.e., $s=2, k=1$). The latter system can be written in the form:

$$\frac{\partial}{\partial t_2} \Phi_1 = (\partial^2 + 2u_1) \Phi_1 \quad , \quad \frac{\partial}{\partial \bar{t}_2} \Phi_1 = [-\bar{\partial}^2 + 2\bar{\partial}(\partial^{-1}(\Phi_1 \Psi_1))] \Phi_1 \quad (11)$$

$$\frac{\partial}{\partial t_2} \Psi_1 = -(\partial^2 + 2u_1) \Psi_1 \quad , \quad \frac{\partial}{\partial \bar{t}_2} \Psi_1 = [\bar{\partial}^2 - 2\bar{\partial}(\partial^{-1}(\Phi_1 \Psi_1))] \Psi_1 \quad (12)$$

where $\bar{\partial} \equiv \partial/\partial \bar{t}_1$. Introducing new notations: $Q \equiv u_1 - 2(\Phi_1 \Psi_1) - \bar{\partial}(\partial^{-1}(\Phi_1 \Psi_1))$; $-iT = t_2 - \bar{t}_2$, $X = t_1 + \bar{t}_1$, $Y = t_1 - \bar{t}_1$, the system (11)–(12) acquires the following form:

$$i\partial_T \Phi_1 = (\partial_X^2 + \partial_Y^2) \Phi_1 + 2(\Phi_1 \Psi_1) \Phi_1 + Q \Phi_1 \quad , \quad (\partial_X^2 - \partial_Y^2) Q + 4\partial_X^2(\Phi_1 \Psi_1) = 0 \quad (13)$$

$$-i\partial_T \Psi_1 = (\partial_X^2 + \partial_Y^2) \Psi_1 + 2(\Phi_1 \Psi_1) \Psi_1 + Q \Psi_1 \quad (14)$$

which is nothing but the standard Davey-Stewartson (DS) system [2]¹. Thus, we succeeded to express solutions of DS system in terms of a pair of (adjoint) EF's of ordinary one-component KP hierarchy supplemented with two additional “ghost” symmetry flows ($\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial \bar{t}_2}$ in the notations of (8)–(9)).

Consider now the τ -function of \mathcal{L} (2) and let us act with $\frac{\partial}{\partial \bar{t}_s}$ on both sides of (3) obtaining $\frac{\partial}{\partial \bar{t}_s} \ln \tau = -\sum_{j=1}^s \partial^{-1}(\Phi_{s-j+1} \Psi_j)$ where we used (8) as well as the t_r -flow eqs. $\frac{\partial}{\partial t_r} \mathcal{M}_s = [\mathcal{L}_+^r, \mathcal{M}_s]_-$. With the help of the well-known recurrence relation for the Schur polynomials: $sp_s(-[\bar{\partial}]) = \sum_{k=1}^s \left(-\frac{\partial}{\partial \bar{t}_k}\right) p_{s-k}(-[\bar{\partial}])$, we find the following important identities [6, 11] relating the tau-function with the “ghost” symmetry generating (adjoint) EF's (here $s, k, j = 1, 2, \dots$):

$$\frac{p_s(-[\bar{\partial}]) (\Phi_1 \tau)}{\tau} = \Phi_{s+1} \quad , \quad \frac{p_s([\bar{\partial}]) (\Psi_1 \tau)}{\tau} = \Psi_{s+1} \quad ; \quad \partial^{-1}(\Phi_k \Psi_j) = \sum_{l=0}^{j-1} \frac{p_{k+l}(-[\bar{\partial}]) p_{j-l+1}([\bar{\partial}]) \tau}{\tau} \quad (15)$$

2. Double-KP System and Its Equivalence to Two-Component KP Hierarchy

In refs.[6, 11] we have shown that the “ghost” symmetry flows from Eqs.(8)-(10) admit their own Lax representation in terms of a $\bar{D} \equiv \partial/\partial \bar{x} \equiv \partial/\partial \bar{t}_1$ pseudo-differential Lax operator $\bar{\mathcal{L}}$ w.r.t.

¹The fact that DS system is contained within the “ghost” symmetry enhanced scalar KP hierarchy (1)–(2),(8)–(9) was first pointed out in [10].

multi-time $(\bar{t}) \equiv (\bar{t}_1 \equiv \bar{x}, \bar{t}_2, \dots)$:

$$\bar{\mathcal{L}} \equiv \bar{W} \bar{D} \bar{W}^{-1} = \bar{D} + \sum_{i=1}^{\infty} \bar{u}_i \bar{D}^{-i} \quad , \quad \frac{\partial \bar{\mathcal{L}}}{\partial \bar{t}_s} = [\bar{\mathcal{L}}_+^s, \bar{\mathcal{L}}] \quad (16)$$

$$\bar{W} = 1 + \sum_{j=1}^{\infty} \frac{p_j(-[\bar{\partial}])\tau(t, \bar{t})}{\tau(t, \bar{t})} \bar{D}^{-j} \quad , \quad \frac{p_j(-[\bar{\partial}])\tau}{\tau} = \partial^{-1}(\Phi_j \Psi_1) \quad (17)$$

where Φ_j, Ψ_1 are the “ghost” symmetry generating (adjoint) EF’s of the original KP hierarchy (1), which we denote as KP_1 . Accordingly, we will denote the “ghost” KP system (16)–(17) as KP_2 . Let us stress that the tau-function $\tau = \tau(t, \bar{t})$ of the KP_1 hierarchy (1)–(2), with time evolution parameters (t) and $(\bar{t}) = fixed$, is simultaneously tau-function of KP_2 hierarchy (16)–(17) with time evolution parameters (\bar{t}) and $(t) = fixed$.

Next, we have found an infinite system of (adjoint) EF’s $\{\bar{\Phi}_k, \bar{\Psi}_k\}_{k=1}^{\infty}$ of KP_2 system (i.e., obeying Eqs.(4) with all quantities replaced with the “barred” ones) :

$$\bar{\Phi}_k = \frac{p_{k-1}(-[\partial]) (\Psi_1 \tau)}{\tau} \quad , \quad \bar{\Psi}_k = \frac{p_{k-1}([\partial]) (\Phi_1 \tau)}{\tau} \quad (18)$$

which generate “ghost” symmetry flows for KP_2 (16) analogous to (8)–(10) such that the corresponding “ghost” symmetry flow parameters coincide with the isospectral flow parameters (t) of the original KP_1 system (1). In particular, we note from Eqs.(18) that $\bar{\Phi}_1 = \Psi_1$ and $\bar{\Psi}_1 = \Phi_1$. Also, for any generic (adjoint) EF’s F, F^* of KP_1 the SEP functions:

$$\bar{F} \equiv \partial^{-1}(F \Psi_1) \quad , \quad \bar{F}^* \equiv \partial^{-1}(\Phi_1 F^*) \quad (19)$$

are, respectively, an EF and adjoint EF of KP_2 [6, 11].

Both KP_1 (original KP hierarchy) together with KP_2 (“ghost” symmetry flows’ KP hierarchy) form a new larger hierarchy called *double-KP* [6, 11] possessing the property of “duality” symmetry, i.e., symmetry under interchanging the rôles of KP_1 and KP_2 . It is defined as follows:

$$(t) \equiv \binom{(1)}{t} \quad , \quad (\bar{t}) \equiv \binom{(2)}{\bar{t}} \quad , \quad \binom{(\alpha)}{t} = \binom{(\alpha)}{t_1, t_2, \dots} \quad , \quad \mathcal{L} \equiv \mathcal{L}^{(1)} \quad , \quad \bar{\mathcal{L}} \equiv \bar{\mathcal{L}}^{(2)} \quad (20)$$

$$\mathcal{L} = W D W^{-1} \quad , \quad W = \sum_{j=0}^{\infty} \frac{p_j(-[\partial])\tau}{\tau} D^{-j} \quad , \quad D \equiv \partial / \partial t_1 \quad (21)$$

$$\Phi_j(t, \bar{t}) \equiv \binom{(12)}{\Phi_j} \binom{(1)(2)}{t, \bar{t}} \quad , \quad \Psi_j(t, \bar{t}) \equiv \binom{(12)}{\Psi_j} \binom{(1)(2)}{t, \bar{t}} \quad , \quad \bar{\Phi}_j(t, \bar{t}) \equiv \binom{(21)}{\bar{\Phi}_j} \binom{(1)(2)}{t, \bar{t}} \quad , \quad \bar{\Psi}_j(t, \bar{t}) \equiv \binom{(21)}{\bar{\Psi}_j} \binom{(1)(2)}{t, \bar{t}} \quad (22)$$

$$\binom{(\alpha\beta)}{\Phi_j} = \varepsilon_{\alpha\beta} \frac{p_{j-1}(-[\partial])\tau_{\alpha\beta}}{\tau} \quad , \quad \binom{(\alpha\beta)}{\Psi_j} = \varepsilon_{\beta\alpha} \frac{p_{j-1}([\partial])\tau_{\beta\alpha}}{\tau} \quad , \quad \varepsilon_{\alpha\beta} = \pm 1 \quad \text{for } \alpha \leq \beta \quad , \quad \alpha > \beta \quad (23)$$

where in (23) we have introduced new tau-functions:

$$\tau_{\alpha\beta} = \varepsilon_{\alpha\beta} \tau \binom{(\alpha\beta)}{\Phi_1} = \varepsilon_{\beta\alpha} \tau \binom{(\beta\alpha)}{\Psi_1} \quad (24)$$

Here the indices (α, β) are taking values $\alpha, \beta = 1, 2$, $\alpha \neq \beta$, and Eqs.(15),(18) have been taken into account.

Accordingly, Eqs.(1),(16),(4),(8)–(10) and their “duals” (for $\bar{\Phi}_k, \bar{\Psi}_k$) can be rewritten in a manifestly “duality”-symmetric form:

$$\partial_s^{(\alpha)(\alpha)} \mathcal{L} = \left[\left(\mathcal{L} \right)_+^{(\alpha)s}, \mathcal{L} \right], \quad \partial_s^{(\beta)(\alpha)} \mathcal{L} = \left[\mathcal{M}_s, \mathcal{L} \right], \quad \partial_s^{(\alpha)} \equiv \frac{\partial}{\partial t_s^{(\alpha)}}, \quad \mathcal{M}_s \equiv \sum_{j=1}^s \Phi_{s-j+1}^{(\alpha\beta)} D^{-1} \Psi_j^{(\alpha)(\alpha\beta)} \quad (25)$$

$$\partial_s^{(\alpha)(\alpha\beta)} \Phi_k = \left(\mathcal{L} \right)_+^{(\alpha)s} (\Phi_k)^{(\alpha\beta)}, \quad \partial_s^{(\alpha)(\alpha)} F = \left(\mathcal{L} \right)_+^{(\alpha)s} (F)^{(\alpha)}; \quad \partial_s^{(\alpha)(\alpha\beta)} \Psi_k = - \left(\mathcal{L}^* \right)_+^{(\alpha)s} (\Psi_k)^{(\alpha\beta)}, \quad \partial_s^{(\alpha)(\alpha)} F^* = - \left(\mathcal{L}^* \right)_+^{(\alpha)s} (F^*)^{(\alpha)} \quad (26)$$

$$\partial_s^{(\beta)(\alpha\beta)} \Phi_k = \sum_{j=1}^s \Phi_{s-j+1}^{(\alpha\beta)} \partial^{-1} \left(\Psi_j \Phi_k \right)^{(\alpha\beta)(\alpha\beta)} - \Phi_{k+s}^{(\alpha\beta)}, \quad \partial_s^{(\beta)(\alpha\beta)} \Psi_k = \sum_{j=1}^s \Psi_j \partial^{-1} \left(\Phi_{s-j+1} \Psi_k \right)^{(\alpha\beta)(\alpha\beta)} + \Psi_{k+s}^{(\alpha\beta)} \quad (27)$$

$$\partial_s^{(\beta)(\alpha)} F = \sum_{j=1}^s \Phi_{s-j+1}^{(\alpha\beta)} \partial^{-1} \left(\Psi_j F \right)^{(\alpha\beta)(\alpha)}, \quad \partial_s^{(\beta)(\alpha)} F^* = \sum_{j=1}^s \Psi_j \partial^{-1} \left(\Phi_{s-j+1} F^* \right)^{(\alpha\beta)(\alpha)} \quad (28)$$

where again $\alpha, \beta = 1, 2$, $\alpha \neq \beta$, $\partial^{-1} \equiv (\partial_1)^{-1}$ and $s = 1, 2, \dots$. In (28) and (26) F, F^* denote generic (adjoint) EF's of \mathcal{L} , i.e., such that they do not belong to the sets $\{\Phi_k, \Psi_k\}$.

In [11] we have shown the equivalence of double-KP hierarchy with the two-component (matrix) Sato KP hierarchy [1], originally formulated within the matrix pseudo-differential Lax formalism, which can be equivalently described by three tau-functions $\tau_{11}, \tau_{12}, \tau_{21}$ depending on two sets of multi-time variables $(t), (\bar{t})$ and obeying the 2×2 matrix Hirota bilinear identities (see Eq.(29) below). The proof proceeds by identifying two-component KP tau-functions with the tau-functions of double-KP hierarchy $\tau_{11} = \tau$, $\tau_{12} = \tau \Phi_1$, $\tau_{21} = -\tau \Psi_1$ as in (21)–(24).

3. Generalization to Multi-Component KP Hierarchies

Let us turn our attention to arbitrary N -component matrix KP hierarchies [1] ($N \geq 2$). They can be equivalently characterized by the set of Hirota bilinear identities:

$$\sum_{\gamma=1}^N \varepsilon_{\alpha\gamma} \varepsilon_{\beta\gamma} \int dz z^{\delta_{\alpha\gamma} + \delta_{\beta\gamma} - 2} e^{\xi(t - \bar{t}', z)} \tau_{\alpha\gamma}(\dots, t - [z^{-1}], \dots) \tau_{\gamma\beta}(\dots, \bar{t}' + [z^{-1}], \dots) = 0 \quad (29)$$

for $N(N-1) + 1$ tau-functions $\tau_{\alpha\alpha} \equiv \tau$ and $\tau_{\alpha\beta}$ ($\alpha \neq \beta$), where now the indices $\alpha, \beta, \gamma = 1, \dots, N$, $\delta_{\alpha\beta}$ are Kronecker symbols and $\varepsilon_{\alpha\beta}$ are the same as in (22). Also, we are using the standard notation $\xi(t, z) \equiv \sum_{l=1}^{\infty} t_l z^l$. Let us recall that (29) contain the following interesting systems of non-linear equations:

$$\partial_1^{(\alpha)(\beta)} \partial_1 \ln \tau = \frac{\tau_{\alpha\beta}}{\tau} \frac{\tau_{\beta\alpha}}{\tau}, \quad \partial_1^{(\gamma)} \left(\frac{\tau_{\alpha\beta}}{\tau} \right) = \varepsilon_{\alpha\beta} \varepsilon_{\alpha\gamma} \varepsilon_{\gamma\beta} \frac{\tau_{\alpha\gamma}}{\tau} \frac{\tau_{\gamma\beta}}{\tau}, \quad \alpha \neq \beta \neq \gamma \quad (30)$$

the second one being the so called N' -wave system ($N' = N(N-1)/2$).

Our main statement is that N -component KP hierarchy defined by (29) is equivalent to the *multiple-KP* hierarchy defined by Eqs.(21),(23)–(28) where now the indices α, β take values $\alpha, \beta = 1, \dots, N$. In other words, this multiple-KP hierarchy consists of N ordinary one-component KP hierarchies KP_α ($\alpha = 1, \dots, N$) given by Lax operators $\mathcal{L}^{(\alpha)}$ (21) in different spaces and generating isospectral flows $\partial_s^{(\alpha)}$ w.r.t. different sets of evolution parameters $(t^{(\alpha)})$, such that the flows $\partial_s^{(\alpha)}$ act on

the rest of KP subsystems KP_β ($\beta \neq \alpha$) as “ghost” symmetry flows. In particular, Eqs.(28) apply also for $F = \Phi_k$ and $F^* = \Psi_k$ with $\gamma \neq \beta \neq \alpha$:

$$\partial_s^{(\beta)(\alpha\gamma)} \Phi_k = \sum_{j=1}^s \Phi_{s-j+1}^{(\alpha\beta)} \partial^{-1} \left(\Psi_j^{(\alpha)} \Phi_k^{(\alpha\gamma)} \right) \quad , \quad \partial_s^{(\beta)(\alpha\gamma)} \Psi_k = \sum_{j=1}^s \Psi_j^{(\alpha\beta)} \partial^{-1} \left(\Phi_{s-j+1}^{(\alpha)} \Psi_k^{(\alpha\gamma)} \right) \quad (31)$$

since Φ_k, Ψ_k are generic (adjoint) EF’s of $\mathcal{L}^{(\alpha)}$ w.r.t. “ghost” symmetry flows $\partial_s^{(\beta)}$ when $\gamma \neq \beta \neq \alpha$.

The detailed proof follows closely the pattern of the proof for the $N = 2$ component KP case [11], making heavy use of the spectral representation identities (5)–(6), and will be given in a separate paper.

Here we will only present an additional important property of multiple-KP system (21),(23)–(28),(31), which appears only in the $N \geq 3$ cases. Namely, for any $\alpha \neq \beta \neq \gamma$ the following identities hold among (adjoint) EF’s:

$$\partial_1^{(\gamma)(\alpha\beta)} \Phi_k = \Phi_1^{(\alpha\gamma)} \Phi_k^{(\gamma\beta)} \quad , \quad \partial_1^{(\gamma)(\alpha\beta)} \Psi_k = \Psi_1^{(\alpha\gamma)} \Psi_k^{(\gamma\beta)} \quad (32)$$

In particular, taking $k=1$ in the first Eq.(32) and taking into account (24), we see that the latter coincides with the N' -wave system (30). In other words, the N' -wave system is reformulated entirely in terms of EF’s of an underlying “ghost” symmetry enhanced ordinary KP hierarchy. Moreover, it is easy to check that (32) are compatible with the “ghost” flow Eqs.(31), and in fact (32) are equivalent to (31).

4. Darboux-Bäcklund Orbits of Multi-Component KP Hierarchies

We will consider DB transformations in their form appropriate for the Sato formulation of KP hierarchies [9]. Namely, DB transformations are pseudo-differential operator “gauge” transformations of the pertinent Lax operators given in terms of (adjoint) EF’s (4).

Let us temporarily return to the simpler case of double-KP hierarchy. In our construction a very instrumental rôle will be played by the following *non-standard* orbit of successive DB transformations for the original KP_1 system (1) ($\mathcal{L} \equiv \mathcal{L}(n)$, $\Phi_j \equiv \Phi_j \equiv \Phi_j^{(n)}$, etc.) :

$$\mathcal{L}(n+1) = T(n)\mathcal{L}(n)T^{-1}(n) \quad , \quad T(n) = \Phi_1 D \Phi_1^{-1} \equiv \Phi_1^{(n)} D \Phi_1^{(n)-1} \quad (33)$$

$$\Phi_l^{(n+1)} = \Phi_1^{(n)} \partial \left(\frac{\Phi_{l+1}^{(n)}}{\Phi_1^{(n)}} \right) \quad , \quad l \geq 1 ; \quad \Psi_1^{(n+1)} = \frac{1}{\Phi_1^{(n)}} \quad , \quad \Psi_j^{(n+1)} = -\frac{1}{\Phi_1^{(n)}} \partial^{-1} \left(\Phi_1^{(n)} \Psi_{j-1}^{(n)} \right) \quad , \quad j \geq 2 \quad (34)$$

$$F^{(n+1)} = \Phi_1^{(n)} \partial \left(\frac{F^{(n)}}{\Phi_1^{(n)}} \right) \quad , \quad F^{*(n+1)} = -\frac{1}{\Phi_1^{(n)}} \partial^{-1} \left(\Phi_1^{(n)} F^{*(n)} \right) \quad (35)$$

for transformations in “positive” direction, as well as adjoint DB transformations, *i.e.*, transformations in “negative” direction:

$$\mathcal{L}(n-1) = \widehat{T}^{*-1}(n)\mathcal{L}(n)\widehat{T}^*(n) \quad , \quad \widehat{T}(n) = \Psi_1 D \Psi_1^{-1} \equiv \Psi_1^{(n)} D \Psi_1^{(n)-1} \quad (36)$$

$$\Phi_1^{(n-1)} = \frac{1}{\Psi_1^{(n)}} \quad , \quad \Phi_l^{(n-1)} = \frac{1}{\Psi_1^{(n)}} \partial^{-1} \left(\Psi_1^{(n)} \Phi_{l-1}^{(n)} \right) \quad , \quad l \geq 2 ; \quad \Psi_j^{(n-1)} = -\Psi_1^{(n)} \partial \left(\frac{\Psi_{j+1}^{(n)}}{\Psi_1^{(n)}} \right) \quad , \quad j \geq 1 \quad (37)$$

$$F^{(n-1)} = \frac{1}{\Psi_1^{(n)}} \partial^{-1} \left(\Psi_1^{(n)} F^{(n)} \right) \quad , \quad F^{*(n-1)} = -\Psi_1^{(n)} \partial \left(\frac{F^{*(n)}}{\Psi_1^{(n)}} \right) \quad (38)$$

In what follows, the DB “site” index (n) will be skipped for brevity whenever this would not lead to ambiguities. Note that under the above DB transformations, the tau-function $\tau \equiv \tau^{(n)}$ transforms as: $\tau^{(n+1)} = \Phi_1 \tau \equiv \Phi_1^{(n)} \tau^{(n)}$, $\tau^{(n-1)} = -\Psi_1 \tau \equiv -\Psi_1^{(n)} \tau^{(n)}$.

Remark. Let us stress the non-canonical form of the (adjoint) DB transformations (34),(37) on the “ghost” symmetry generating (adjoint) EF’s. On the other hand, for generic (adjoint) EF’s F, F^* the (adjoint) DB transformations (35),(38) read as usual [9].

The crucial property of the above DB orbit of the original one-component KP_1 hierarchy is that it induces an orbit of DB transformations for the whole double-KP system (21)–(28). More precisely, as shown in [6, 11], “ghost” symmetries (8) commute with DB transformations (33)–(38) of the KP_1 hierarchy, and induce the following DB transformations on the “ghost” KP_2 hierarchy (recall from Eqs.(18), that $\bar{\Phi}_1^{(n)} = \Psi_1^{(n)}$, $\bar{\Psi}_1^{(n)} = \Phi_1^{(n)}$) :

$$\bar{\mathcal{L}}(n+1) = \left(\frac{1}{\bar{\Psi}_1^{(n)}} \bar{D}^{-1} \bar{\Psi}_1^{(n)} \right) \bar{\mathcal{L}}(n) \left(\frac{1}{\bar{\Psi}_1^{(n)}} \bar{D} \bar{\Psi}_1^{(n)} \right) \quad , \quad \bar{\mathcal{L}}(n-1) = \left(\bar{\Phi}_1^{(n)} \bar{D} \frac{1}{\bar{\Phi}_1^{(n)}} \right) \bar{\mathcal{L}}(n) \left(\bar{\Phi}_1^{(n)} \bar{D}^{-1} \frac{1}{\bar{\Phi}_1^{(n)}} \right) \quad (39)$$

$$\bar{\Phi}_j^{(n-1)} = \bar{\Phi}_1^{(n)} \bar{\partial} \left(\frac{\bar{\Phi}_{j+1}^{(n)}}{\bar{\Phi}_1^{(n)}} \right) \quad , \quad j \geq 1 \quad ; \quad \bar{\Psi}_1^{(n-1)} = \frac{1}{\bar{\Phi}_1^{(n)}} \quad , \quad \bar{\Psi}_l^{(n-1)} = -\frac{1}{\bar{\Phi}_1^{(n)}} \bar{\partial}^{-1} \left(\bar{\Phi}_1^{(n)} \bar{\Psi}_{l-1}^{(n)} \right) \quad , \quad l \geq 2 \quad (40)$$

$$\bar{F}^{(n-1)} = \bar{\Phi}_1^{(n)} \bar{\partial} \left(\frac{\bar{F}^{(n)}}{\bar{\Phi}_1^{(n)}} \right) \quad , \quad \bar{F}^{*(n-1)} = -\frac{1}{\bar{\Phi}_1^{(n)}} \bar{\partial}^{-1} \left(\bar{\Phi}_1^{(n)} \bar{F}^{*(n)} \right) \quad (41)$$

$$\bar{\Phi}_1^{(n+1)} = \frac{1}{\bar{\Psi}_1^{(n)}} \quad , \quad \bar{\Phi}_l^{(n+1)} = \frac{1}{\bar{\Psi}_1^{(n)}} \bar{\partial}^{-1} \left(\bar{\Psi}_1^{(n)} \bar{\Phi}_{l-1}^{(n)} \right) \quad , \quad l \geq 2 \quad ; \quad \bar{\Psi}_j^{(n+1)} = -\bar{\Psi}_1^{(n)} \bar{\partial} \left(\frac{\bar{\Psi}_{j+1}^{(n)}}{\bar{\Psi}_1^{(n)}} \right) \quad , \quad j \geq 1 \quad (42)$$

$$\bar{F}^{(n+1)} = \frac{1}{\bar{\Psi}_1^{(n)}} \bar{\partial}^{-1} \left(\bar{\Psi}_1^{(n)} \bar{F}^{(n)} \right) \quad , \quad \bar{F}^{*(n+1)} = -\bar{\Psi}_1^{(n)} \bar{\partial} \left(\frac{\bar{F}^{*(n)}}{\bar{\Psi}_1^{(n)}} \right) \quad (43)$$

where \bar{F}, \bar{F}^* are generic (adjoint) EF’s of $\bar{\mathcal{L}}$. For each DB “site” (n) the corresponding Lax operators and (adjoint) EF’s from (33)–(43) define a double-KP system as in (21)–(28). Therefore, the DB orbit (33)–(43) defines an orbit of DB transformations for the associated two-component (matrix) KP hierarchy.

One can prove a similar statement also for generic DB transformations of KP_1 hierarchy:

$$\tilde{\mathcal{L}} = \left(F D F^{-1} \right) \mathcal{L} \left(F D^{-1} F^{-1} \right) \quad , \quad \tilde{\Phi}_j = F \partial \left(\frac{\Phi_j}{F} \right) \quad , \quad \tilde{\Psi}_j = -\frac{1}{F} \partial^{-1} (F \Psi_j) \quad (44)$$

$$\hat{\mathcal{L}} = \left(F^{*-1} D^{-1} F^* \right) \hat{\mathcal{L}} \left(F^{*-1} D F^* \right) \quad , \quad \hat{\Phi}_j = \frac{1}{F^*} \partial^{-1} (F^* \Phi_j) \quad , \quad \hat{\Psi}_j = -F^* \partial \left(\frac{\Psi_j}{F^*} \right) \quad (45)$$

where F, F^* are generic (adjoint) EF’s of \mathcal{L} , namely, the “ghost” symmetries (8) commute with generic DB transformations (44)–(45).

Remark. Comparing (33)–(35) with (39),(42)–(43), and (36)–(38) with (39)–(41), we find that DB transformations of KP_1 hierarchy w.r.t. Φ_1 corresponds to adjoint-DB transformations of KP_2 hierarchy w.r.t. $\bar{\Psi}_1$, and vice versa.

Let us now go back to the general N -component KP hierarchy. Picking up any pair KP_α and KP_β (with $\alpha \neq \beta$) of one-component KP subsystems of multiple-KP hierarchy (25)–(28) and (31), we can construct a DB orbit w.r.t. $\Phi_1^{(\alpha\beta)}$ for this pair as in (33)–(35), (40), (42)–(43) by literally repeating the above construction with the identifications: $\mathcal{L} \equiv \mathcal{L}^{(\alpha)}$, $\bar{\mathcal{L}} \equiv \mathcal{L}^{(\beta)}$, $\Phi_j \equiv \Phi_j^{(\alpha\beta)}$, $\Psi_j \equiv \Psi_j^{(\alpha\beta)}$, $F \equiv \Phi_k^{(\alpha\gamma)}$, $F^* \equiv \Psi_k^{(\alpha\gamma)}$, $\bar{\Phi}_j \equiv \Phi_j^{(\beta\alpha)}$, $\bar{\Psi}_j \equiv \Psi_j^{(\beta\alpha)}$, $\bar{F} \equiv \Phi_k^{(\beta\gamma)}$, $\bar{F}^* \equiv \Psi_k^{(\beta\gamma)}$, where $\gamma \neq \alpha \neq \beta$. Hence such DB orbit, which we will call $DB_{(\alpha\beta)}$ orbit, preserves the whole multiple-KP hierarchy. Also, according to the last Remark in Section 3, $DB_{(\alpha\beta)}$ orbit is equivalent to adjoint- $DB_{(\beta\alpha)}$ orbit, i.e., the DB orbit w.r.t. $\Psi_1^{(\beta\alpha)}$. Then, a general DB orbit preserving multiple-KP hierarchy, or equivalently, N -component matrix KP hierarchy, is obtained by combining the $DB_{(\alpha\beta)}$ orbits for all pairs (α, β) with $\alpha \neq \beta$.

We are particularly interested in DB orbits passing through the “free” N -component KP hierarchy, i.e., with $\mathcal{L} = D^{(\alpha)}$ and hence $\tau = const$ in (21). Then one can easily show that the general DB orbit $DB_{(12,13,\dots,1N)}$ for the N -component KP hierarchy passing through the “free” one is built up from a union of $DB_{(1\beta)}$ orbits of the form (34)–(35), (42)–(43) with $\beta = 2, \dots, N$. Accordingly, it is labelled by the set of integers $(n_{12}, n_{13}, \dots, n_{1N}; m)$ indicating n_{12} DB iterations w.r.t. $\Phi_1^{(12)}$ or $|n_{12}|$ adjoint-DB iterations w.r.t. $\Psi_1^{(12)}$ for $n_{12} < 0$, n_{13} DB iterations w.r.t. $\Phi_1^{(13)}$ or $|n_{13}|$ adjoint-DB iterations w.r.t. $\Psi_1^{(13)}$ for $n_{13} < 0$, etc., whereas the last integer indicates m steps of DB transformations w.r.t. generic EF’s $F_1^{(1)}, \dots, F_m^{(1)}$ of KP_1 subsystem, i.e., such that they do not belong to any of the “ghost” symmetry generating sets of $\Phi_k^{(1\beta)}$.

5. Darboux-Bäcklund Solutions

Taking into account relations (24) and the structure of the $DB_{(1\beta)}$ suborbits ($\beta = 2, \dots, N$) of the form (34)–(35), (42)–(43) allows us to express all non-diagonal tau-functions of the N -component KP hierarchy on any site $(n_{12}, n_{13}, \dots, n_{1N}; m)$ of the full orbit $DB_{(12,13,\dots,1N)}$ in terms of DB shifts of the diagonal tau-function as follows (from now on we will use the shortened notations $n_{1\alpha} \equiv n_\alpha$):

$$\tau_{1\alpha}^{(\dots, n_\alpha, \dots)} = \eta_\alpha \tau^{(\dots, n_\alpha + 1, \dots)} \quad , \quad \tau_{\alpha 1}^{(\dots, n_\alpha, \dots)} = -\eta_\alpha \tau^{(\dots, n_\alpha - 1, \dots)} \quad (46)$$

$$\tau_{\alpha\beta}^{(\dots, n_\alpha, \dots, n_\beta, \dots)} = \eta_\alpha \eta_\beta \tau^{(\dots, n_\alpha - 1, \dots, n_\beta + 1, \dots)} \quad , \quad \tau_{\beta\alpha}^{(\dots, n_\alpha, \dots, n_\beta, \dots)} = \eta_\alpha \eta_\beta \tau^{(\dots, n_\alpha + 1, \dots, n_\beta - 1, \dots)} \quad (47)$$

where $\eta_\alpha \equiv \text{sign}(n_\alpha)$ and $1 < \alpha < \beta \leq N$. Since τ is the tau-function of scalar KP_1 sub-hierarchy, the problem of finding the complete DB solution of the full N -component KP hierarchy is reduced to applying the well-known techniques of (adjoint) DB iterations in ordinary one-component KP hierarchy within the Sato/tau-function formulation. Using the above techniques and taking into account the specific form of DB orbits (34)–(35) and (37)–(38), we obtain (assuming explicitly that part of the DB iterations are adjoint-DB ones) the following Wronskian-type expression for the diagonal tau-function:

$$\tau^{(-n_2, \dots, -n_k, n_{k+1}, \dots, n_N; m)} / \tau^{(0, \dots, 0; 0)} = (-1)^{\sum_{j=2}^k n_j - 1} \times \quad (48)$$

$$\widetilde{W} \left[\begin{array}{cccccccc} (1, k+1) & (1, k+1) & (1N) & (1N) & (1) & (1) & (12) & (12) & (1k) & (1k) \\ \Phi_1 & \Phi_{n_{k+1}} & \Phi_1 & \Phi_{n_N} & F_1 & F_m & \Psi_1 & \Psi_{n_2} & \Psi_1 & \Psi_{n_k} \end{array} \right]$$

with the short-hand notation for Wronskian-type determinants:

$$\widetilde{W} [f_1, \dots, f_k; f_1^*, \dots, f_l^*] \equiv \det \left\| \begin{array}{cc} \partial^{a-1} f_b & \partial^{a-1} f_{k-l+d} \\ \partial^{-1} (f_b f_c^*) & \partial^{-1} (f_{k-l+d} f_c^*) \end{array} \right\|, \quad a, b=1, \dots, k-l; c, d=1, \dots, l \quad (49)$$

Recalling property (19) and the “ghost” symmetry flow Eqs.(27),(10) and (32), we find that we can replace the SEP’s in (48) (cf. (49)) as follows:

$$\partial^{-1} \left(\begin{array}{c} (1\beta) \\ \Phi_{i_\beta} \end{array} \begin{array}{c} (1\alpha) \\ \Psi_1 \end{array} \right) = \begin{array}{c} (\alpha\beta) \\ \Phi_{i_\beta} \end{array}, \quad \partial^{-1} \left(\begin{array}{c} (1) \\ F_b \end{array} \begin{array}{c} (1\alpha) \\ \Psi_1 \end{array} \right) = \begin{array}{c} (\alpha) \\ F_b \end{array}, \quad \partial^{-1} \left(\begin{array}{c} (1) \\ F_b \end{array} \begin{array}{c} (1\alpha) \\ \Psi_{j_\alpha} \end{array} \right) = \partial^{j_\alpha-1} \begin{array}{c} (\alpha) \\ F_b \end{array} + \dots \quad (50)$$

where $\alpha = 2, \dots, k$, $j_\alpha = 1, \dots, n_\alpha$, $\beta = k+1, \dots, N$, $i_\beta = 1, \dots, n_\beta$ and $b = 1, \dots, m$, and where the dots in second Eqs.(50) indicate terms with lower derivatives on the corresponding EF’s which cancel in the determinant. Therefore, the Wronskian-type determinant on the r.h.s. of Eq.(48) together with (50) acquires the form of a *multiple* Wronskian, generalizing the double Wronskians of the first ref.[12]. Let us recall that all entries in the multiple Wronskian (48)–(50) are (derivatives of) EF’s and adjoint EF’s of the respective initial KP_α subsystems (26). In the case of “free” initial

point on the DB orbit we have: $\partial_s F_b = (\partial_1)^s F_b$, $\partial_s \Phi_i = (\partial_1)^s \Phi_i$ and $\Phi_i = (-\partial_1)^{i-1} \Phi_1$.

As an example let us consider the general DB solution (48) in the case $N = 2$ and with a “free” initial $\tau^{(0;0)} = 1$:

$$\tau^{(-n;m)} = (-1)^{n-1} \det \left\| \begin{array}{cc} \partial^{a-1} F_b & \partial^{a-1} F_{m-n+d} \\ \bar{\partial}^{c-1} \bar{F}_b & \bar{\partial}^{c-1} \bar{F}_{m-n+d} \end{array} \right\|, \quad a, b=1, \dots, m-n; c, d=1, \dots, n \quad (51)$$

where F_b, \bar{F}_b are arbitrary free EF’s of KP_1 and KP_2 one-component sub-hierarchies of two-component KP hierarchy, respectively: $F_b = \int d\lambda \varphi_b^{(-)}(\lambda) e^{\xi(t, \lambda)}$. The tau-functions (51), taking into account (24), provide the following series of Wronskian solutions for DS system (13):

$$Q^{(-n;m)} = 4\partial_X^2 \ln \tau^{(-n;m)}, \quad \Phi_1^{(-n;m)} = \frac{\tau^{(-n+1;m)}}{\tau^{(-n;m)}}, \quad \Psi_1^{(-n;m)} = \frac{\tau^{(-n-1;m)}}{\tau^{(-n;m)}} \quad (52)$$

In the particular case $m = 2n$ and taking special forms of the spectral densities of the pertinent EF’s in (51) $\varphi_b^{(-)}(\lambda) = \sum_{a=1}^{2n} c_{ba}^{(-)} \delta(\lambda - \lambda_a)$ with constant coefficients $c_{ba}^{(-)}$, the series (52) with (51) contains the well-known n^2 (multi-)dromion solutions [13] in the double-Wronskian form given in the first ref.[12] (after making appropriate choice for the constant parameters in order to ensure reality properties).

More detailed analysis of the new series of multiple-Wronskian solutions (46)–(50) of N -component KP hierarchies, in particular, how other known soliton-type solutions are fitting there, will be given elsewhere².

6. Conclusions

In the present note we have shown that, given an ordinary one-component KP hierarchy KP_1 , we can always construct a N -component matrix KP hierarchy, embedding the original one, in the following way. We choose $N-1$ infinite sets $\left\{ \begin{array}{c} (1\alpha) \\ \Phi_k, \Psi_k \end{array} \right\}_{k=1}^\infty$, $\alpha = 2, \dots, N$, of (adjoint) EF’s of the initial

²Let us note that our multiple-Wronskian DB orbit differs from the DB orbit generated via matrix EF’s [9] within Sato matrix pseudo-differential operator formulation. The latter cannot be written compactly in a Wronskian form.

KP_1 which we use to construct an infinite-dimensional abelian algebra of additional (“ghost”) symmetries (25)–(28),(31). The one-component KP hierarchy equipped with such additional symmetry structure turns out to be equivalent to the standard N -component matrix KP hierarchy. Namely, with the help of a subset of the “ghost” symmetry generating (adjoint) EF’s of KP_1 we can define new tau-functions $\tau_{\alpha\beta} = \tau \varepsilon_{\alpha\beta} \overset{(1)}{\partial}^{-1} \begin{pmatrix} (1\beta) \\ \Phi_1 \end{pmatrix} \begin{pmatrix} (1\alpha) \\ \Psi_1 \end{pmatrix}$ (cf. (24),(32)) with $\alpha \neq \beta$, $\alpha, \beta = 1, \dots, N$, which together with τ (the tau-function of the initial KP_1 hierarchy) satisfy Hirota bilinear identities of N -component matrix KP hierarchy (29).

Furthermore, we have shown that there exists a special non-standard Darboux-Bäcklund orbit of the initial KP_1 hierarchy (Section 4), which preserves the above mentioned additional (“ghost”) symmetry structure and which thereby generates DB transformations of the whole N -component matrix KP hierarchy. This fact allows to use the well-known DB techniques within the context of the ordinary scalar KP subsystem KP_1 to generate soliton-like tau-function solutions of all higher N -component KP hierarchies which take the form of multiple Wronskians (Section 5). In particular, we find in this way new series of multiple-Wronskian solutions to well-known systems of integrable nonlinear soliton equations contained within the N -component matrix KP hierarchy: Davey-Stewartson system (for $N = 2$, containing the previously obtained dromion solutions) and N' -wave system (for $N \geq 3$).

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